Probabilistic Engineering Mechanics 59 (2020) 103018

Contents lists available at ScienceDirect

Probabilistic Engineering Mechanics

journal homepage: www.elsevier.com/locate/probengmech

A multilevel moving particles method for reliability estimation

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ABSTRACT

A multilevel moving particles method is developed for the estimation of failure probabilities that are computed from numerical approximations of the performance function. The algorithm balances the statistical error of the estimate and the approximation error. This is achieved by computing a telescoping sum of estimates for the difference in the number of moves at subsequent levels of the approximation error. The sample variance of these estimates decreases with decreasing approximation error. Thus, the number of samples that has to be evaluated with high accuracy is reduced compared to a single level computation. Therefore, the proposed algorithm is very efficient for structural reliability analysis problems, in which highly accurate evaluations of the performance function are necessary and require a huge computational effort.

1. Introduction

Markov Chain Monte Carlo simulation methods allow estimating small failure probabilities efficiently, even for problems that involve a high-dimensional vector of input random variables [1]. Subset simulation can be considered as the most prominent method in this class. In subset simulation, the failure probability is computed as the telescoping product of larger probabilities that require sampling from conditional distributions. Recently, a generalization of subset simulation in the sense of particle methods has been proposed [2], where the value of the performance function is associated to each sample, samples are moved to new positions in the design space and the number of moves for the initial samples to reach the failure domain are counted and yield an estimator for the failure probability, which is of comparable accuracy and efficiency as the subset simulation estimator [3]. The algorithm allows for an easy parallel implementation. Just as for subset simulation, sampling from conditional distributions is required when moving a particle.

In most practical applications, the performance function is approximated numerically to a certain level h. In order to obtain an efficient simulation algorithm, it is necessary to balance the statistical error and the approximation error. Here, a multilevel moving particles simulation method is proposed that balances both errors by computing a telescoping sum of estimates for the number of moves. For each term in the telescoping sum, it is necessary to compute as corrector the difference of the number of moves for each initial sample with two consecutive accuracy levels using the same random numbers in the Markov Chain Monte Carlo simulation.

For the multilevel moving particles method, the sample variance decreases with decreasing approximation error. Thus, the number of samples that has to be evaluated with high accuracy is reduced compared to a single level computation. Therefore, the proposed algorithm is very efficient for problems where highly accurate evaluations of the performance function are necessary and require a tremendous computational effort.

The paper is organized as follows: in the next section, the moving particles algorithm is introduced and briefly compared with subset simulation. Following this, the idea of multilevel simulation is explained and applied to the moving particles algorithm. The multilevel moving particles algorithm is then tested on examples comprising a simple cumulative distribution function, a stochastic ordinary differential equation and a stochastic partial differential equation. Finally, conclusions are drawn.

2. Moving particles algorithm

Denote the failure domain by $F = \{\boldsymbol{\theta} \in \mathbb{R}^n | g(\boldsymbol{\theta}) < 0\}$, where $g(\boldsymbol{\theta})$ is the performance function and $\boldsymbol{\theta}$ is a vector of random variables. The probability of failure is given by the integral

$$P_F = \int_F p(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}, \tag{1}$$

where $p(\boldsymbol{x})$ denotes the joint probability density function of the random vector $\boldsymbol{\theta}$.

The moving particles algorithm yields an estimate of P_F. It starts with an initial Monte Carlo simulation (MCS) with N_m samples. These initial samples are then moved to the failure domain by the following procedure: The values $g(\boldsymbol{\theta}_j)$, $j = 1, \dots, N$, of the N samples are ranked. The sample with the maximum value of the performance function is moved: a Markov chain Monte Carlo simulation (MCMC) is carried out starting from one of the remaining samples and the final state of the Markov chain is accepted, if the value of the performance function could be reduced. Otherwise, the sample is simply replaced by the initial value of the Markov chain. The Markov chain can be generated either by application of the classical Metropolis–Hastings algorithm or by direct sampling from a normal transition kernel.

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https://doi.org/10.1016/j.probengmech.2020.103018
Received 21 January 2020; Accepted 27 January 2020
Available online 5 February 2020

For each initial sample, the number M of moves until it reaches the failure domain is count. As has been shown in [2], the number of moves to get an initial sample into the failure domain follows a Poisson distribution with parameter $\lambda = -\log P_F$. The estimator for the parameter of the Poisson distribution is obtained from $\lambda = E[M]$ as

$$\hat{\lambda} = \frac{\sum_{j=1}^{N} M_j}{N}, \tag{2}$$

where M_j, $j = 1\ldots, N$ denotes the number of moves until the initial sample j reaches the failure domain.

In order to obtain an unbiased estimate, it is mandatory that the trajectories of the Poisson process generated from the initial samples remain independent until the samples finally reach the failure domain. In [4], two means are proposed to maintain the independence:

- Burn-in: The Markov chain simulation is carried out with a burn-in period. The burn-in should ensure the independence of the initial and the final state of the Markov chain.
- Seed avoidance: Repeated use of the same initial state for the Markov chain should be avoided. Once a sample has been used as initial state, the sample and its offsprings should not be used as initial state again.

The coefficient of variation for the failure probability estimated with the moving particles algorithm is given by

$$\delta_{mp} = \sqrt{\frac{-\log P_F}{N}}, \tag{3}$$

cf. [2], and the average number of function evaluations is

$$N_{mp} = N(1 - T \log P_F), \tag{4}$$

where the first term accounts for the initial Monte Carlo simulation and the second term for the Markov chain samples (with burn-in period T).

A parallel version of the algorithm is easily obtained, if the k samples with highest values of the performance function are moved in parallel.

The moving particles algorithm can be considered as subset simulation with a maximum number of subsets. Thus, for each subset, only one sample is discarded and resampled by a Markov chain that takes as initial value one of the retained samples. However, there are several differences with respect to the original subset simulation algorithm:

- In subset simulation, the number of steps is rather small; however, the number of steps in the moving particles algorithm is maximal.
- In subset simulation, only p_0 (usually 10%) of the samples are retained in each step and serve as seed for the Markov chains. In contrast, only one sample is resampled in each step of the moving particles algorithm, and the seed can be selected among the other samples.
- The moving particles algorithm requires that all initial samples finally reach the failure domain.
- The moving particles algorithm has a mathematical foundation in Poisson process theory.

By equating (4) and the corresponding equation for subset simulation with fixed conditional probability p_0 of the subsets,

$$N_{sub} = N_s \left(1 + (1 - p_0)\frac{\log P_F}{\log p_0}\right), \tag{5}$$

both algorithms can be compared. For a common choice of $p_0 = 0.1$, the coefficient of variation of both algorithms is nearly the same for $N = 0.1 N_s$ [3]. From Eqs. (4) and (5), one obtains the relationship

$$\frac{N_{sub}}{N_{mp}} = \frac{10(1 - 0.39 \log P_F)}{1 - T \log P_F} \tag{6}$$

By setting this expression to one, a burn-in period T can be obtained as a function of the failure probability for which both algorithms would require approximately the same amount of function evaluations.

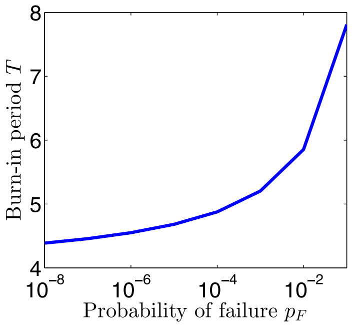

Fig. 1. Comparison of subset simulation algorithm and moving particles algorithm. Burn-in period, such that the number of performance function evaluations of subset simulation and the moving particles algorithm is approximately the same.

This relationship is depicted in Fig. 1 a). It can be shown that the obtained burn-in period (4 to 5 steps of the Markov chain) is in the range of values that is sufficient to obtain independence in practical applications [3] and thus the number of function evaluations for both algorithms is of the same order of magnitude. For small failure probabilities, subset simulation becomes slightly more efficient than the moving particles algorithm, while for larger probabilities of failure, the opposite is the case.

3. Multilevel simulation

In general, the performance function $g(\boldsymbol{\theta})$ is not known exactly, but is computed by numerical approximations. The approximations rely on a discretization and can be characterized by a discretization parameter h; the numerically computed performance function being denoted by $g_h(\boldsymbol{\theta})$. In what follows, it is assumed that it is possible to compute a finite series of approximations obtained from a series of decreasing values h_l, $l = 0, \ldots, L$, for the discretization parameter h. The index l is called level of the approximation. A typical example for these approximations would be solutions with different time stepsizes, computational grids or meshes. Often, the discretization parameters can be related by $h_l = 2^{-l} h_0$ for a given initial discretization parameter h_0.

Denote by Q_h a quantity of interest that depends on $g_h(\boldsymbol{\theta})$. A multilevel estimator for such a quantity of interest is obtained from the telescoping sum [5,6]

$$E[Q_{h_L}] = E[Q_{h_0}] + \sum_{l=1}^{L} E[Q_{h_l} - Q_{h_{l-1}}]. \tag{7}$$

The aim is to compute each of the estimates on the right-hand side of this equation individually. A reduction of the overall computational effort can be expected from the fact that the variance of the differences decreases to zero with increasing level l and thus, estimates of the contributions from highly accurate performance function require less samples. Moreover, the fact that there is a nested sequence of approximations might be beneficial similar as for multigrid methods.

It is important to note that in the estimation of $E[Q_{h_l} - Q_{h_{l-1}}]$ both Q_{h_l} and $Q_{h_{l-1}}$ are evaluated for the same samples. If the dimension of the random vector depends on the discretization parameter h, it is necessary to generate the samples for the finer scale and to obtain the corresponding samples on the coarser scale by coarse-graining.

For reliability estimation, an extension of multilevel Monte Carlo simulation has been proposed in [7], where a selective refinement strategy has been applied, such that realizations far away from the critical value $g(\boldsymbol{\theta}) = 0$ are solved to a lower accuracy than those close to the critical value, which further reduces the computational effort.

In [8], a multilevel estimator is combined with subset simulation. However, when passing to a new subset while changing the accuracy of the approximation for the performance function, the subsets might not be nested anymore. This then leads to the necessity of a burn-in period for the Markov chain and an additional Markov chain simulation is needed for each subset.

4. Multilevel moving particles algorithm

For the moving particles algorithm, the quantity of interest is the number of moves to reach the failure domain. Denote by M_h the number of moves when the numerically computed performance function $g_h(\boldsymbol{\theta})$ is applied. The multilevel estimator for the Poisson distribution parameter at level L and thus for $-\log P_{F_{h_L}}$, where $P_{F_{h_L}}$ denotes the failure probability computed with the approximated performance function $g_{h_L}(\boldsymbol{\theta})$ at approximation level L is obtained from

$$E[M_{h_L}] = E[M_{h_0}] + \sum_{l=1}^{L} E[M_{h_l} - M_{h_{l-1}}] \tag{8}$$

and reads

$$\hat{M}_{h_L}^{ML} = \frac{1}{N_0}\sum_{i=1}^{N_0} M_{h_0}^{(i)} + \sum_{l=1}^{L}\frac{1}{N_l}\sum_{i=1}^{N_l}(M_{h_l}^{(i)} - M_{h_{l-1}}^{(i)}). \tag{9}$$

The corresponding single level estimator is

$$\hat{M}_{h_L}^{SL} = \frac{1}{N_{SL}}\sum_{i=1}^{N_{SL}} M_{h_L}^{(i)}. \tag{10}$$

The mean square error for the single level estimator is

$$\begin{aligned} e(M_{h_L}^{SL})^2 &= V(M_{h_L}^{SL}) + \left(E[M_{h_L}^{SL}] - E[M]\right)^2 \\ &= N_{SL}^{-1} V(M_{h_L}) + \left(E[M_{h_L}] - E[M]\right)^2 \end{aligned} \tag{11}$$

where $V(.)$ denotes the variance operator. The corresponding error for the multilevel estimator is

$$e(M_{h_L}^{ML})^2 = \sum_{l=0}^{L} N_l^{-1} V(Y_l) + \left(E[M_{h_L}] - E[M]\right)^2, \tag{12}$$

where $Y_0 = M_{h_0}$ and $Y_l = M_{h_l} - M_{h_{l-1}}$, $l > 0$. In both expressions, the last term is the numerical approximation error, while the first term is the statistical error. For a given overall precision $\epsilon > 0$ such that $e(.)^2 < \epsilon^2$, we wish to achieve a statistical as well as an approximation error less than $\epsilon^2/2$. For the approximation error, we have

$$\left(E[M_{h_L}] - E[M]\right)^2 < \frac{\epsilon^2}{2}. \tag{13}$$

If

$$\left|\log P_F - \log P_{F_{h_l}}\right| = \mathcal{O}(h_l), \tag{14}$$

we obtain the condition $h_l = \mathcal{O}(\epsilon)$ for the approximation error. For the statistical error of the single level estimator, we impose

$$N_{SL}^{-1} V(M_{h_L}) = \mathcal{O}(\epsilon^2), \tag{15}$$

which leads, if $V(M_{h_L})$ is approximately constant, to $N_{SL} = \mathcal{O}(\epsilon^{-2})$ for the number of samples, the total number of function estimations being thus $N_{SL}(1 - T\log P_{F_{h_L}})$, cf. Eq. (4). Suppose that the computational cost $C(M_{h_l}^{(i)}) = (1 - T\log P_{F_{h_l}})$ to compute a single sample is of order $\mathcal{O}(h_l^{-r})$ for some $r > 0$. Then the computational cost for the single level estimator is of order

$$C(M_L^{SL}) = \mathcal{O}(N_{SL} h_L^{-r}) = \mathcal{O}(\epsilon^{-2-r}). \tag{16}$$

For the multilevel estimator, the total computational cost is

$$C(M_L^{ML}) = \sum_{l=0}^{L} N_l C(Y_l^{(i)}). \tag{17}$$

Fixing the total computational cost, the statistical error becomes minimal, if $N_l = \lambda\sqrt{V(Y_l)/C(Y_l^{(i)})}$, with

$$\lambda = \epsilon^{-2}\sum_{l=0}^{L}\sqrt{V(Y_l)C(Y_l^{(i)}} \tag{18}$$

if the statistical error should be equal to ϵ^2. In this case, the total computational cost of the multilevel algorithm is

$$C(M_L^{ML}) = \epsilon^{-2}\left(\sum_{l=0}^{L}\sqrt{V(Y_l)C(Y_l^{(i)})}\right)^2. \tag{19}$$

If Eq. (14) holds and

$$\left|\log P_F^2 - \log P_{F_{h_l}}^2\right| = \mathcal{O}(h_l), \tag{20}$$

then from the fact that M_{h_l} follows a Poisson distribution, the variance of Y_l is of $\mathcal{O}(h_l)$. Depending on the increase of the costs $C(M_{h_l}^{(i)})$ with h_l, which is described by the exponent r, the first or the last term in the sum in (19) will dominate. In fact, if $0 < r < 1$, the costs increase less than he variance will decrease and thus the sum in (19) is dominated by $\sqrt{V(Y_0)C(Y_0^{(i)})}$, so that $C(M_L^{ML}) = \mathcal{O}(\epsilon^{-2})$. On the other hand, if $r > 1$, then the sum is dominated by $\sqrt{V(Y_L)C(Y_L^{(i)})}$ and $C(M_L^{ML}) = \mathcal{O}(\epsilon^{-2}h_L h_L^{-r}) = \mathcal{O}(\epsilon^{-1-r})$.

Thus, compared with the computational cost of order $\mathcal{O}(\epsilon^{-2-r})$ for the single level estimator, considerable savings are obtained with the multilevel estimator. This can be attributed to the fact that the order of magnitude for the cost of the single level estimator involves the product of the variance $V(Y_0)$ and the cost $C(Y_L^{(i)})$.

Conditions (14) and (20) depend directly on the approximation of the performance function. This is shown by the following lemma.

Proposition 1. *If* $|g_h(x) - g(x)| < |g_h(x)|$ *or* $|g_h(x) - g(x)| \le h$*, then* $\left|P_F - P_{F_h}\right| < h$.

Proof. [7], Lemma 3.4. □

Lemma 1. *If* $\left|P_F - P_{F_h}\right| < h$*, then*

1. $\left|\log P_F - \log P_{F_h}\right| < C_1 h$
2. $\left|\log {P_F}^2 - \log {P_{F_h}}^2\right| < C_2 h$

for $h \to 0$.

Proof.

1. If $P_F < P_{F_h}$, then

$$\log P_{F_h} - \log P_F = \log\frac{P_{F_h}}{P_F} < \frac{P_{F_h}}{P_F} - 1 = \frac{P_{F_h} - P_F}{P_F} < \frac{h}{P_F} \tag{21}$$

On the other hand, if $P_F > P_{F_h} > 0$, one has

$$\log P_F - \log P_{F_h} = \log\frac{P_F}{P_{F_h}} < \frac{P_F}{P_{F_h}} - 1 = \frac{P_F - P_{F_h}}{P_{F_h}} < \frac{h}{P_F}\frac{P_F}{P_{F_h}} < \frac{h}{P_F}\frac{1}{1 - \frac{h}{P_F}} \tag{22}$$

and thus for $h < \frac{P_F}{2}$

$$\log P_F - \log P_{F_h} < \frac{2h}{P_F} \tag{23}$$

Therefore $\left|\log P_F - \log P_{F_h}\right| < C_1 h$ with $C_1 = \frac{2}{P_F}$.

2. If $P_F < P_{F_h}$, then

$$\log {P_F}^2 - \log {P_{F_h}}^2 = \log\frac{P_{F_h}}{P_F}(-(\log P_F + \log P_{F_h})) < \frac{h}{P_F}(-(2\log P_F)). \tag{24}$$

and if $P_F > P_{F_h}$, then for $h < \frac{P_F}{2}$:

$$\begin{aligned} \log {P_{F_h}}^2 - \log {P_F}^2 &= \log \frac{P_F}{P_{F_h}}(-(\log P_F + \log P_{F_h})) \\ &< \frac{2h}{P_F}(-(2\log P_F) + \log P_F - \log P_{F_h}) \\ &< \frac{2h}{P_F}(-(2\log P_F) + \frac{2h}{P_F}). \end{aligned} \tag{25}$$

So $\left|\log {P_F}^2 - \log {P_{F_h}}^2\right| < C_2 h$ with $C_2 = 2(-(2\log P_F) + \frac{2h}{P_F})/P_F$. □

Remark. Both C_1 and C_2 are rather large.

The result for the computational cost of the multilevel moving particles estimator can thus be summarized as follows:

Theorem 1. *If $\{h_l\}$, $l = 0 \ldots L$, is a finite decreasing series such that $\left|g_{h_l}(x) - g(x)\right| < \left|g_{h_l}(x)\right|$ or $\left|g_{h_l}(x) - g(x)\right| \leq h_l$ and $C[M_{h_l}] = \mathcal{O}(h_l^{-r})$, then for a given error bound $\epsilon > 0$, there are values N_l, $l = 1, \ldots, L$, for which the root mean-square error of the multilevel estimator obtained from (8) is less than ϵ and the computational cost is of order*

$$C(M_L^{ML}) = \begin{cases} \mathcal{O}(\epsilon^{-2}), & r < 1 \\ \mathcal{O}(\epsilon^{-2}(\log \epsilon)^2), & r = 1 \\ \mathcal{O}(\epsilon^{-1-r}), & r > 1 \end{cases} \tag{26}$$

Proof. [6], Theorem 3.1. □

5. Examples

5.1. Standard normal distribution

The first example deals with a standard normally distributed random variable X and the performance function $g(X) = X - y$, such that the probability of failure is $P_F = 1 - \Phi(y)$, where $\Phi(y)$ denotes the cumulative distribution function of X. The performance function is perturbed by noise and the amplitude of the noise away from the threshold y is increased so that the condition in Proposition 1 is satisfied.

For this simple example, the convergence of the mean and the variance of M_{h_l} and the difference $M_{h_l} - M_{h_{l-1}}$ will be studied. The result for the variance is shown in Fig. 2 (a). It can be seen that the convergence follows indeed a power law for the multilevel estimator, while it is independent of the levels for the single level estimator. The same holds for the mean, cf. Fig. 2 (b).

5.2. Stochastic heat equation with random heat source

Consider the linear stochastic partial differential equation

$$\mathrm{d}u = \frac{\partial^2 u}{\partial x^2}\mathrm{d}t + \theta u \mathrm{d}t + \sigma \mathrm{d}W_t \tag{27}$$

with parameters θ and σ, where dW are the increments of a standard cylindrical Wiener process

$$W_t = \sum_{k=1}^{\infty} W_t^k \sin(k\pi x), \tag{28}$$

and W_t^k are standard independent Brownian motions.

The boundary conditions are $u(x = 0, t) = u(x = 1, t) = 0$ and the initial condition reads

$$u(x, 0) = \sqrt{2}\sum_{k=1}^{\infty} \sin(k\pi x), \tag{29}$$

so that the solution of the stochastic partial differential equation decouples and can be written as

$$u(x, t) = \sum_{k=1}^{\infty} u_k(t)\sin(k\pi x), \tag{30}$$

where the coefficients $u_k(t)$ are described by the linear stochastic ordinary differential equations

$$\mathrm{d}u_k = (-\pi^2 k^2 + \theta)u_k \mathrm{d}t + \sigma \mathrm{d}W_t^k. \tag{31}$$

Their solutions at time t are

$$u_k(t) = \exp((-\pi^2 k^2 + \theta)t) + \xi_t^k, \tag{32}$$

where ξ_t^k is a normal random variable with mean zero and variance

$$\frac{\sigma^2(1 - \exp(2(-\pi^2 k^2 + \theta)t))}{2(\pi^2 k^2 - \theta)}. \tag{33}$$

Failure is assumed to occur if $u(x = 0.5, t = T)$ is larger than a given threshold.

The solution of the stochastic partial differential equation (27) is approximated by truncating the solution at $k = k_{max}$ and the approximate solution is simulated. The approximation levels are defined by different values for the parameter k_{max}. The value of the approximation parameter h is computed by comparing the exact and the approximate failure probability.

Fig. 3 shows the result for $\theta = 1$, $\sigma = 1$, $T = 1/50$ and a threshold value of 1.5. It can be seen that both the mean value and the variance decrease with decreasing discretization parameter. However, for smaller values of the approximation parameters, the reduction of the mean value and particularly of the variance is less pronounced.

5.3. Stochastic heat equation with random thermal conductivity

Consider the diffusion equation

$$\frac{\partial u}{\partial t} = \alpha(x, \theta)\frac{\partial^2 u}{\partial x^2}, \tag{34}$$

where $\alpha(x, \theta)$ is a normal random field described by the following truncated Karhunen–Loève expansion

$$\alpha(x, \theta) = 1 + \sum_{i=1}^{M} \sqrt{\lambda_i} f_i(x)\xi_i(\theta), \tag{35}$$

where $\xi(\theta) = [\xi_1(\theta), \xi_2(\theta), \ldots, \xi_M(\theta)]$ are independent standard normal random variables and λ_i, $f_i(x)$, $i = 1, \ldots, M$ are the eigenvalues and eigenvectors of a Fredholm integral equation of 2nd kind for the exponential covariance kernel

$$Cov(x_1, x_2) = \sigma^2 \exp(-\|x_1 - x_2\|). \tag{36}$$

The boundary conditions are given by $u(x = 0, t) = 0$ and $u(x = \ell, t) = 0$, the initial condition is $u(x, t = 0) = \sin(\frac{\pi x}{\ell})$.

For the polynomial chaos expansion

$$u(x, t, \xi) = \sum_{i=0}^{P} u_i(x, t)\psi_i(\xi) \tag{37}$$

of the solution, cf. [9], the following partial differential equation governs the expansion coefficients:

$$\sum_{i=0}^{P} \frac{\partial u_i}{\partial t}\psi_i(\xi) = (1 + \sum_{i=1}^{M} \sqrt{\lambda_i} f_i(x)\xi_i(\theta))(\sum_{i=0}^{P} \frac{\partial^2 u_i}{\partial x^2}\psi_i(\xi)). \tag{38}$$

Application of a Galerkin scheme with respect to the polynomial chaos and a finite difference approximation leads to the equation

$$\begin{aligned} \frac{1}{\Delta t}\sum_{i=0}^{P}\sum_{j=0}^{P} b_{ij}(u_j^{(m,n+1)} - u_j^{(m,n)}) = {} & \frac{1}{(\Delta x)^2}\sum_{i=0}^{P}\sum_{j=0}^{P} b_{ij} \\ & \times (u_j^{(m+1,n)} - 2u_j^{(m,n)} + u_j^{(m-1,n)}) \\ & + \frac{1}{(\Delta x)^2}\sum_{i=0}^{P}\sum_{j=0}^{P}\sum_{k=1}^{M} d_{ijk}\sqrt{\lambda_k} f_k(x) \\ & \times (u_j^{(m+1,n)} - 2u_j^{(m,n)} + u_j^{(m-1,n)}) \end{aligned} \tag{39}$$

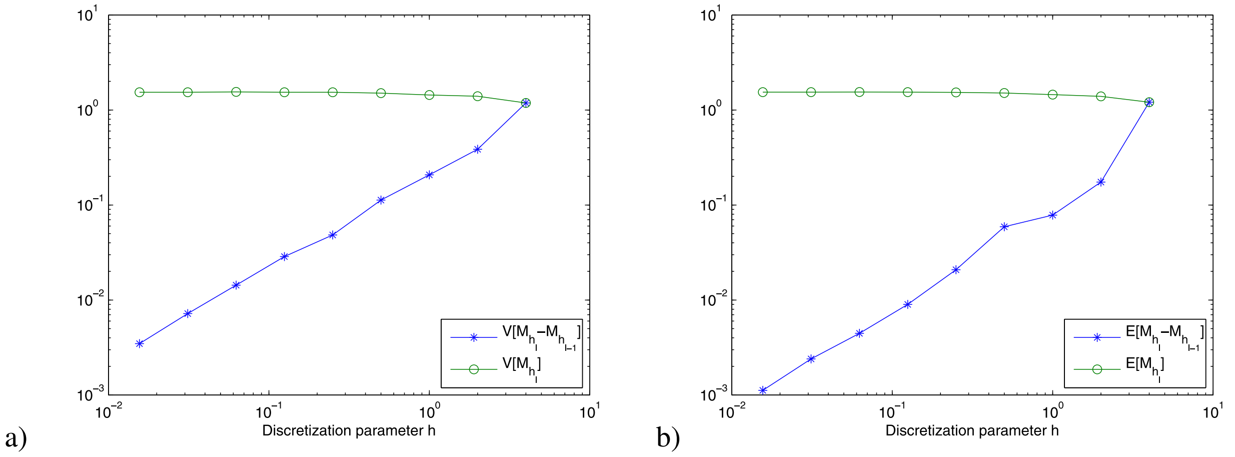

Fig. 2. Variance and mean of M_{h_l} and $M_{h_l} - M_{h_{l-1}}$.

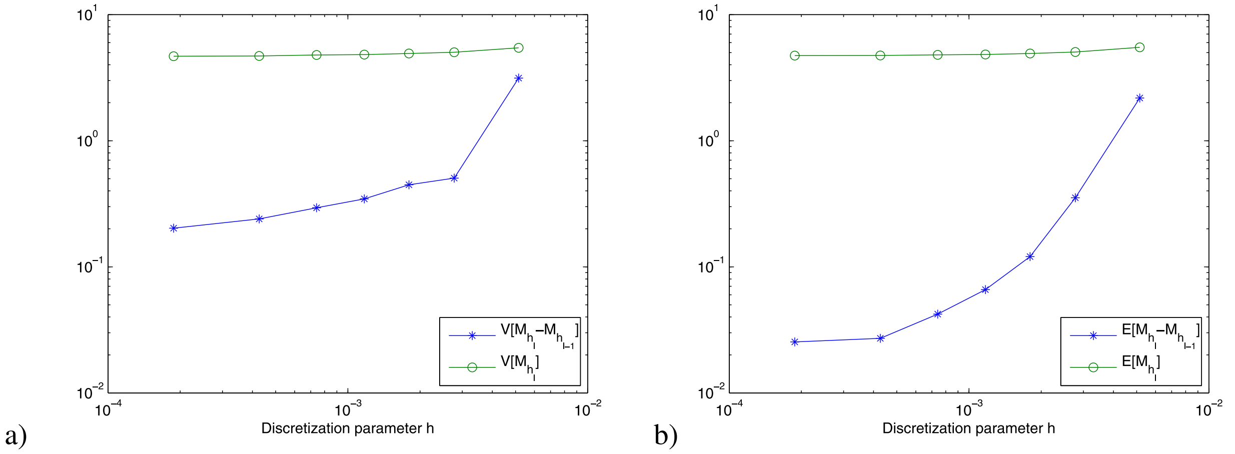

Fig. 3. Variance and mean of M_{h_l} and $M_{h_l} - M_{h_{l-1}}$.

for the discretized expansion coefficients $u_i^{(m,n+1)}$, cf. [10], where

$$\begin{aligned} b_{ij} &= \int \psi_i(x)\psi_j(x)\mathrm{d}P_G(x), \\ d_{ijk} &= \int x_k\psi_i(x)\psi_j(x)\mathrm{d}P_G(x) \end{aligned} \tag{40}$$

and $\mathrm{d}P_G(x)$ denotes the M-dimensional standard normal measure.

Failure is assumed to occur if $u(x = \ell/2, t = T)$ is larger than a given threshold.

Fig. 4 shows the result for $\sigma = 1.$, $\gamma = 0.025$, $\ell = 1$, $T = 1.$, a 4 term Karhunen–Loève expansion, a 4th order polynomial chaos expansion and a threshold value of 1. The results confirm the previous findings, i.e. the mean value and the variance of the multilevel estimator decrease with decreasing discretization parameter. The reduction for the mean value is less pronounced for smaller values of the discretization parameter.

5.4. Burgers' equation

In order to consider a nonlinear problem, the proposed approach is applied to Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \alpha(x,\theta)\frac{\partial^2 u}{\partial x^2}, \tag{41}$$

where the stochastic viscosity $\alpha(x,\theta)$ is given by the truncated Karhunen–Loève expansion (35). The boundary conditions are given by $u(x = 0, t) = 0$ and $u(x = \ell, t) = 0$, the initial condition is $u(x, t = 0) = \sin(\frac{\pi x}{\ell})$.

The following partial differential equation governs the expansion coefficients of the polynomial chaos expansion (37)

$$\sum_{i=0}^{P}\frac{\partial u_i}{\partial t}\psi_i(\xi)+\sum_{i=0}^{P}u_i\psi_i(\xi)\sum_{j=0}^{P}\frac{\partial u_j}{\partial x}\psi_j = (1+\sum_{i=1}^{M}\sqrt{\lambda_i}f_i(x)\xi_i(\theta))(\sum_{i=0}^{P}\frac{\partial^2 u_i}{\partial x^2}\psi_i(\xi)). \tag{42}$$

Application of a Galerkin scheme with respect to the polynomial chaos and a finite difference approximation leads to the equation

$$\begin{aligned} &\frac{1}{\Delta t}\sum_{i=0}^{P}\sum_{j=0}^{P}b_{ij}(u_j^{(m,n+1)} - u_j^{(m,n)}) \\ &\quad + \frac{1}{2(\Delta x)}\sum_{i=0}^{P}\sum_{j=0}^{P}\sum_{k=0}^{P}c_{ijk}u_j^{(m,n)}(u_k^{(m+1,n)} - u_k^{(m-1,n)}) \\ &= \frac{1}{(\Delta x)^2}\sum_{i=0}^{P}\sum_{j=0}^{P}\sum_{k=1}^{M}d_{ijk}\sqrt{\lambda_k}f_k(x) \\ &\quad \times (u_j^{(m+1,n)} - 2u_j^{(m,n)} + u_j^{(m-1,n)}) \\ &\quad + \frac{1}{(\Delta x)^2}\sum_{i=0}^{P}\sum_{j=0}^{P}b_{ij}(u_j^{(m+1,n)} - 2u_j^{(m,n)} + u_j^{(m-1,n)}) \end{aligned} \tag{43}$$

for the discretized expansion coefficients $u_i^{(m,n+1)}$, cf. [10], where $c_{ijk} = \int x_k\psi_i(x)\psi_j(x)\psi_k(x)\mathrm{d}P_G(x)$ and b_{ij} and d_{ijk} are given by (40).

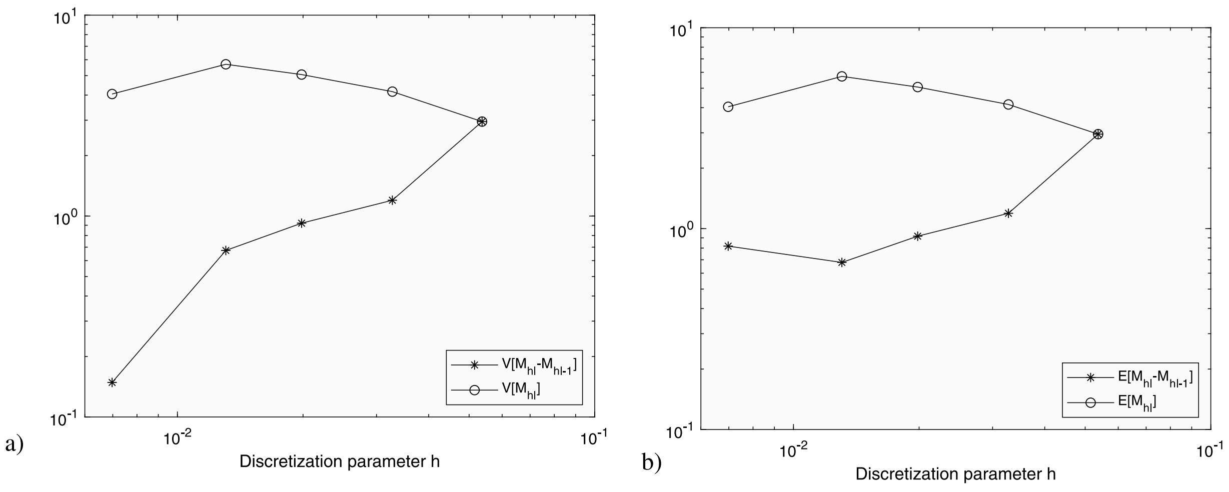

Fig. 4. Variance and mean of M_{h_ℓ} and $M_{h_\ell} - M_{h_{\ell-1}}$.

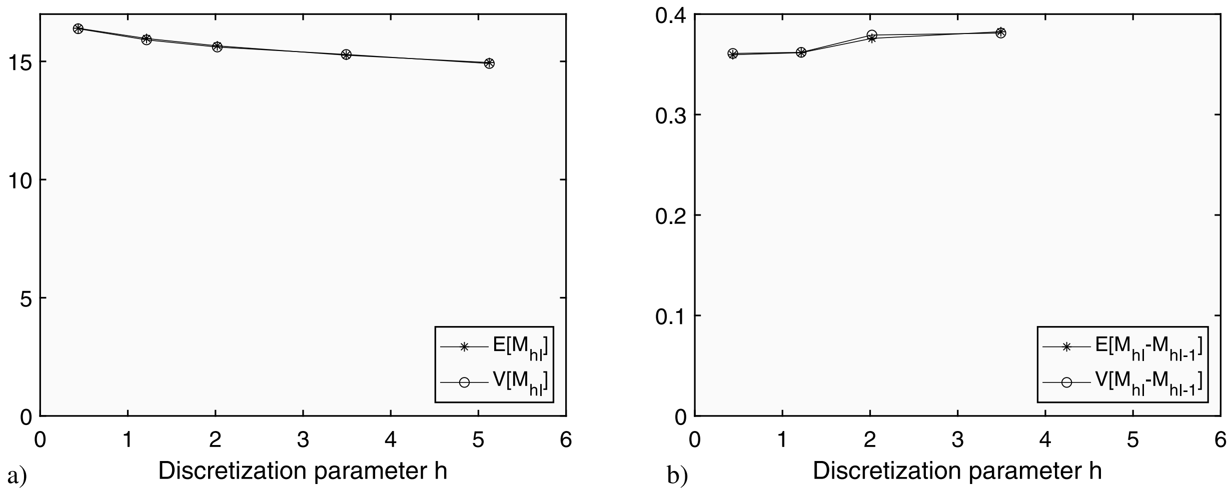

Fig. 5. Mean values and variances for the single levels and the level differences.

Failure is assumed to occur if $u(x = \ell/2, t = T)$ is larger than a given threshold.

Fig. 5 compares the means and variances for the single levels and for the level differences. The parameters of the model were set to $\sigma = 0.5$, $\gamma = 0.001$, $\ell = 1$, $T = 1$. As in the previous examples, a 4 term Karhunen–Loève expansion and a 4th order polynomial chaos expansion were applied. The threshold value was 0.1.

The single level results show that both mean and variance of the number of moves increase with decreasing discretization parameter. Thus, the computed probability of failure decreases with decreasing discretization parameter. In contrast to this, the mean and variance of the differences, which are much smaller than the single level results, decrease with decreasing discretization parameter.

6. Conclusions

In this paper, a multilevel moving particle method for reliability estimation is presented. The method balances the approximation error and the statistical error by computing a telescoping sum of estimates for the number of moves of the particles. It is demonstrated that this approach leads to a considerable reduction of the total computational cost. The approach can be extended by taking the data and model error in a Bayesian setting into account.

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